Indian Statistical Institute
Mid-Semestral examination
Analysis of several variables - MMath I

Max. Marks : 40 Time : 2 hours

State clearly the results that you use. Justify your answers.

(1) (a) When do you say a function $f : U \longrightarrow \mathbb{R}^m$ defined on an open subset U of $\mathbb{R}^n$ is differentiable at $x \in U$. If f is differentiable at $x \in U$, show that the partial derivatives $\frac{\partial f_i}{\partial x_j}(x)$, $1 \leq i \leq m$, $1 \leq j \leq n$, exist and

$$df_x(e_j) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(x) u_i$$

where $e_1, \ldots, e_n$ and $u_1, \ldots, u_m$ denote the standard ordered bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and f_i, $1 \leq i \leq m$ are the coordinate functions of f. [2+6]

(b) Show that the function

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$ and that all directional derivatives exist at $(0,0)$. Is f differentiable at $(0,0)$? [2+2+2]

(c) State the *Inverse function theorem.* Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(t) = \begin{cases} 0 & t = 0 \\ t + 2t^2 \sin\left(\frac{1}{t}\right) & t \neq 0 \end{cases}$$

Show that f is differentiable and that f is not $1-1$ on any interval containing $t = 0$. Explain the connection of the function f with the Inverse function theorem. [2+6+2]

(2) (a) What are *critical points*? Construct a differentiable function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$C = \{(x, x^2) : x \in \mathbb{R}\}$$

as its set of critical points. [2+6]

(b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be of class C^2. Let x be a critical point of f and assume that the $n \times n$ matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)$$

is invertible. Show that there is an open set U containing x that contains no other critical point of f. [8]